Lecture 30

Reciprocity Theorem

Reciprocity theorem is one of the most important theorems in electromagnetics. With it we can develop physical intuition to ascertain if a certain design or experiment is right or wrong. It also tells us what is possible or impossible in the design of many systems. Reciprocity theorem is like “tit-for-tat” relationship in humans: Good-will is reciprocated with good will while ill-will is countered with ill-will. Both Confucius (551 BC–479 BC) and Jesus Christ (4 BC–AD 30) epoused the concept that, “Don’t do unto others that you don’t like others to do unto you.” But in electromagnetics, this beautiful relationship can be expressed precisely and succinctly using mathematics. We shall see how this is done.

Figure 30.1: (Left) A depiction of Confucius from a stone fresco from the Western Han dynasty (202 BC–9 AD). The emphasis of the importance of “reciprocity” by Confucius Analects translated by D. Hinton [175]. (Right) A portrait of Jesus that is truer to its form. Jesus teaching from the New Testament says, “Do unto others as you would have them do unto you.” Luke 6:31 and Matthew 7:12 [176]. The subsequent portraits of these two sages are more humanly urbane.

30.1 Mathematical Derivation

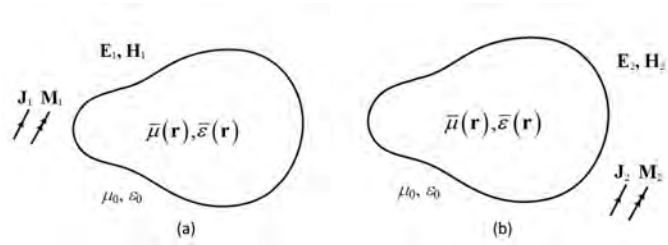

Figure 30.2: The geometry for proving reciprocity theorem. We perform two experiments on the same object or scatterer: (a) With sources $\mathbf{J}_1$ and $\mathbf{M}_1$ turned on, generating fields $\mathbf{E}_1$ and $\mathbf{H}_1$, and (b) With sources $\mathbf{J}_2$ and $\mathbf{M}_2$ turned on, generating fields $\mathbf{E}_2$ and $\mathbf{H}_2$. Magnetic currents, by convention, are denoted by double arrows.

Consider a general anisotropic inhomogeneous medium in the frequency domain where both $\overline{\boldsymbol{\mu}}(\mathbf{r})$ and $\overline{\boldsymbol{\varepsilon}}(\mathbf{r})$ are described by permeability tensor and permittivity tensor over a finite part of space as shown in Figure 30.2. This representation of the medium is quite general, and it can include dispersive and conductive media as well. It can represent complex terrain, or complicated electronic circuit structures in circuit boards or microchips, as well as complicated antenna structures.

We can do a Gedanken experiment[1] where a scatterer or an object is illuminated by fields from two sets of sources which are turned on and off consecutively. This is illustrated in Figure 30.2: When only $\mathbf{J}_1$ and $\mathbf{M}_1$ are turned on, they generate fields $\mathbf{E}_1$ and $\mathbf{H}_1$ in this medium. On the other hand, when only $\mathbf{J}_2$ and $\mathbf{M}_2$ are turned on, they generate $\mathbf{E}_2$ and $\mathbf{H}_2$ in this medium. Therefore, the pertinent equations in the frequency domain, for linear time-invariant systems, for these two cases are[2]

$$\nabla \times \mathbf{E}_1 = -j\omega\overline{\boldsymbol{\mu}} \cdot \mathbf{H}_1 - \mathbf{M}_1 \quad (30.1.1)$$

$$\nabla \times \mathbf{H}_1 = j\omega\overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E}_1 + \mathbf{J}_1 \quad (30.1.2)$$

$$\nabla \times \mathbf{E}_2 = -j\omega\overline{\boldsymbol{\mu}} \cdot \mathbf{H}_2 - \mathbf{M}_2 \quad (30.1.3)$$

$$\nabla \times \mathbf{H}_2 = j\omega\overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E}_2 + \mathbf{J}_2 \quad (30.1.4)$$

We would like to find a simplifying expression for the divergence of the following quantity,

$$\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2) = \mathbf{H}_2 \cdot \nabla \times \mathbf{E}_1 - \mathbf{E}_1 \cdot \nabla \cdot \mathbf{H}_2 \quad (30.1.5)$$

so that the divergence theorem can be invoked. To this end, and from the above, we can show

[1] Thought experiment in German.

[2] The current sources are impressed currents so that they are immutable, and not changed by the environment they are immersed in [50, 172].

that (after left dot-multiply (30.1.1) with $\mathbf{H}_2$ and (30.1.4) with $\mathbf{E}_1$),

$$\mathbf{H}_2 \cdot \nabla \times \mathbf{E}_1 = -j\omega \mathbf{H}_2 \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}_1 - \mathbf{H}_2 \cdot \mathbf{M}_1 \tag{30.1.6}$$

$$\mathbf{E}_1 \cdot \nabla \times \mathbf{H}_2 = j\omega \mathbf{E}_1 \cdot \overline{\varepsilon} \cdot \mathbf{E}_2 + \mathbf{E}_1 \cdot \mathbf{J}_2 \tag{30.1.7}$$

Then, using the above, and the following identity, we get the second equality in the following expression:

$$\begin{aligned}\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2) &= \mathbf{H}_2 \cdot \nabla \times \mathbf{E}_1 - \mathbf{E}_1 \cdot \nabla \times \mathbf{H}_2 \\ &= -j\omega \mathbf{H}_2 \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}_1 - j\omega \mathbf{E}_1 \cdot \overline{\varepsilon} \cdot \mathbf{E}_2 - \mathbf{H}_2 \cdot \mathbf{M}_1 - \mathbf{E}_1 \cdot \mathbf{J}_2\end{aligned} \tag{30.1.8}$$

By the same token,

$$\nabla \cdot (\mathbf{E}_2 \times \mathbf{H}_1) = -j\omega \mathbf{H}_1 \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}_2 - j\omega \mathbf{E}_2 \cdot \overline{\varepsilon} \cdot \mathbf{E}_1 - \mathbf{H}_1 \cdot \mathbf{M}_2 - \mathbf{E}_2 \cdot \mathbf{J}_1 \tag{30.1.9}$$

If one assumes that

$$\overline{\boldsymbol{\mu}} = \overline{\boldsymbol{\mu}}^t, \qquad \overline{\varepsilon} = \overline{\varepsilon}^t \tag{30.1.10}$$

or when the tensors are symmetric, then $\mathbf{H}_1 \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}_2 = \mathbf{H}_2 \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}_1$ and $\mathbf{E}_1 \cdot \overline{\varepsilon} \cdot \mathbf{E}_2 = \mathbf{E}_2 \cdot \overline{\varepsilon} \cdot \mathbf{E}_1$.[3]

Upon subtracting (30.1.8) and (30.1.9), one gets

$$\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) = -\mathbf{H}_2 \cdot \mathbf{M}_1 - \mathbf{E}_1 \cdot \mathbf{J}_2 + \mathbf{H}_1 \cdot \mathbf{M}_2 + \mathbf{E}_2 \cdot \mathbf{J}_1 \tag{30.1.11}$$

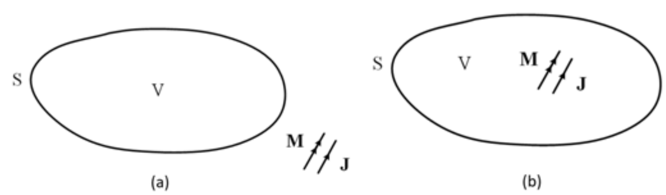

Figure 30.3: The geometry for proving reciprocity theorem when the surface S: (a) does not enclose the sources, and (b) encloses the sources. In the figure, the sources are supposed to be either $(\mathbf{M}_1, \mathbf{J}_1)$ producing fields $(\mathbf{E}_1, \mathbf{H}_1)$ or $(\mathbf{M}_2, \mathbf{J}_2)$ producing fields $(\mathbf{E}_2, \mathbf{H}_2)$.

Now, integrating (30.1.11) over a volume V bounded by a surface S, and invoking Gauss' divergence theorem, we have the reciprocity theorem that

$$\begin{aligned}&\oiint_S d\mathbf{S} \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \\ &\qquad = -\iiint_V dV [\mathbf{H}_2 \cdot \mathbf{M}_1 + \mathbf{E}_1 \cdot \mathbf{J}_2 - \mathbf{H}_1 \cdot \mathbf{M}_2 - \mathbf{E}_2 \cdot \mathbf{J}_1]\end{aligned} \tag{30.1.12}$$

[3]It is to be noted that in matrix algebra, the dot product between two vectors are often written as $\mathbf{a}^t \cdot \mathbf{b}$, but in the physics literature, the transpose on $\mathbf{a}$ is implied. Therefore, the dot product between two vectors is just written as $\mathbf{a} \cdot \mathbf{b}$.

When the volume V contains no sources (see Figure 30.3), the reciprocity theorem reduces to

$$\oiint_S d\mathbf{S} \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) = 0 \tag{30.1.13}$$

The above is also called Lorentz reciprocity theorem by some authors.[4]

Next, when the surface S contains all the sources (see Figure 30.3), then the right-hand side of (30.1.12) will not be zero. On the other hand, when the surface $S \to \infty$, $\mathbf{E}_1$ and $\mathbf{H}_2$ becomes spherical waves which can be approximated by plane waves sharing the same $\boldsymbol{\beta}$ vector. Moreover, under the plane-wave approximation, $\omega\mu_0\mathbf{H}_2 = \boldsymbol{\beta} \times \mathbf{E}_2$, $\omega\mu_0\mathbf{H}_1 = \boldsymbol{\beta} \times \mathbf{E}_1$, then

$$\mathbf{E}_1 \times \mathbf{H}_2 \sim \mathbf{E}_1 \times (\boldsymbol{\beta} \times \mathbf{E}_2) = \mathbf{E}_1(\boldsymbol{\beta} \cdot \mathbf{E}_2) - \boldsymbol{\beta}(\mathbf{E}_1 \cdot \mathbf{E}_2) \tag{30.1.14}$$

$$\mathbf{E}_2 \times \mathbf{H}_1 \sim \mathbf{E}_2 \times (\boldsymbol{\beta} \times \mathbf{E}_1) = \mathbf{E}_2(\boldsymbol{\beta} \cdot \mathbf{E}_1) - \boldsymbol{\beta}(\mathbf{E}_2 \cdot \mathbf{E}_1) \tag{30.1.15}$$

But $\boldsymbol{\beta} \cdot \mathbf{E}_2 = \boldsymbol{\beta} \cdot \mathbf{E}_1 = 0$ in the far field and the $\boldsymbol{\beta}$ vectors are parallel to each other. Therefore, the two terms on the left-hand side of (30.1.12) cancel each other, and it vanishes when $S \to \infty$. (They cancel each other so that the remnant field vanishes faster than $1/r^2$. This is necessary as the surface area S is growing larger and proportional to r^2.)

As a result, when $S \to \infty$, (30.1.12) can be rewritten simply as

$$\int_V dV[\mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{H}_2 \cdot \mathbf{M}_1] = \int_V dV[\mathbf{E}_1 \cdot \mathbf{J}_2 - \mathbf{H}_1 \cdot \mathbf{M}_2] \tag{30.1.16}$$

The inner product symbol is often used to rewrite the above as

$$\langle \mathbf{E}_2, \mathbf{J}_1 \rangle - \langle \mathbf{H}_2, \mathbf{M}_1 \rangle = \langle \mathbf{E}_1, \mathbf{J}_2 \rangle - \langle \mathbf{H}_1, \mathbf{M}_2 \rangle \tag{30.1.17}$$

where the inner product $\langle \mathbf{A}, \mathbf{B} \rangle = \int_V dV \mathbf{A}(\mathbf{r}) \cdot \mathbf{B}(\mathbf{r})$.

The above inner product is also called **reaction**, a concept introduced by Rumsey [177]. The above is also called the **Rumsey reaction theorem**. Sometimes, the above is rewritten more succinctly and tersely as

$$\langle 2, 1 \rangle = \langle 1, 2 \rangle \tag{30.1.18}$$

where

$$\langle 2, 1 \rangle = \langle \mathbf{E}_2, \mathbf{J}_1 \rangle - \langle \mathbf{H}_2, \mathbf{M}_1 \rangle \tag{30.1.19}$$

The concept of inner product or reaction can be thought of as a kind of "measurement". The reciprocity theorem can be stated as that the fields generated by sources 2 as "measured" by sources 1 is equal to fields generated by sources 1 as "measured" by sources 2. This measurement concept is more lucid if we think of these sources as Hertzian dipoles.

[4] Harrington, Time-Harmonic Electric Field [50].

30.2 Conditions for Reciprocity

It is seen that the above proof hinges on (30.1.10). In other words, the anisotropic medium has to be described by symmetric tensors. They include conductive media, but not gyrotropic media which is non-reciprocal. A ferrite biased by a magnetic field is often used in electronic circuits, and it corresponds to a gyrotropic, non-reciprocal medium.[5] Also, our starting equations (30.1.1) to (30.1.4) assume that the medium and the equations are linear time invariant so that Maxwell's equations can be written down in the frequency domain easily.

30.3 Application to a Two-Port Network and Circuit Theory

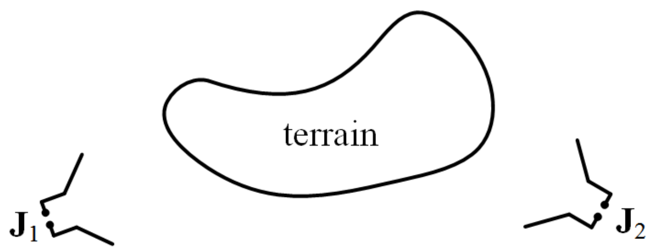

Figure 30.4: A geometry for proving the circuit relationship between two antennas using reciprocity theorem. Circuit relationship is possible when the ports of the antennas are small compared to wavelength.

The reciprocity theorem can be used to distill and condense the interaction between two antennas over a complex terrain as long as the terrain comprises reciprocal media, namely, if $\overline{\boldsymbol{\mu}} = \overline{\boldsymbol{\mu}}^t$ and $\overline{\varepsilon} = \overline{\varepsilon}^t$ for these media.[6] In Figure 30.4, we assume that antenna 1 is driven by impressed current $\mathbf{J}_1$ while antenna 2 is driven by impressed current $\mathbf{J}_2$. It is assumed that the antennas are made from reciprocal media, such as conductive media. Since the system is linear time invariant, it can be written as the interaction between two ports as in circuit theory as shown in Figure 30.5. Assuming that these two ports are small compared to wavelengths, then we can apply circuit concepts like potential theory by letting $\mathbf{E} = -\nabla\Phi$ in the neighborhood of the ports. Thus, we can define voltages and currents at these ports, and V-I relationships can be established in the manner of circuit theory.

[5]Non-reciprocal media are important for making isolators in microwave. Microwave signals can travel from Port 1 to Port 2, but not vice versa.

[6]It is to be noted that a gyrotropic medium considred in Section 9.1 does not safisfy this reciprocity criteria, but it does satisfy the lossless criteria of Section 10.3.2.

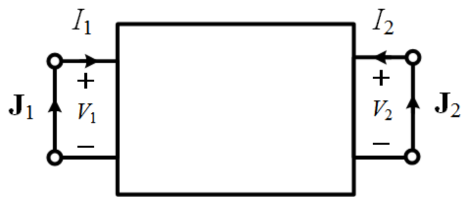

Figure 30.5: The interaction between two antennas in the far field of each other can be reduced to a circuit theory description since the input and output ports of the antennas are small compared to wavelength.

Focusing on a two-port network as shown in Figure 30.5, we have from circuit theory that [178]

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{30.3.1}$$

This form is permissible since we have a linear time-invariant system, and this is the most general way to establish a linear relationship between the voltages and the currents. Furthermore, the matrix elements Z_{ij} can be obtained by performing a series of open-circuit and short-circuit measurements as in circuit theory.

Then assuming that the port 2 is turned on with $\mathbf{J}_2 \neq 0$, while port 1 is turned off with $\mathbf{J}_1 = 0$. In other words, port 1 is open circuit, and the source $\mathbf{J}_2$ is an impressed current source [7] that will produce an electric field $\mathbf{E}_2$ at port 1. Since the current at port 1 is turned off, or that $\mathbf{J}_1 = 0$, the voltage measured at port 1 is the open-circuit voltage V_1^{oc}. Please note here that $\mathbf{J}_1$ and $\mathbf{J}_2$ are impressed currents and are only defined in their respective port. Consequently, the reaction

$$\langle \mathbf{E}_2, \mathbf{J}_1 \rangle = \int_V dV (\mathbf{E}_2 \cdot \mathbf{J}_1) = I_1 \int_{\text{Port 1}} \mathbf{E}_2 \cdot d\mathbf{l} = -I_1 V_1^{oc} \tag{30.3.2}$$

Even though port 1 is assumed to be off, the $\mathbf{J}_1$ is the impressed current to be used above is the $\mathbf{J}_1$ when port 1 is turned on. Given that the port is in the circuit physics regime, we assume the currents in wire to be constant, then the current $\mathbf{J}_1$ is a constant current at the port when it is turned on. Or the current I_1 can be taken outside the integral. In slightly more details, the current $\mathbf{J}_1 = \hat{l} I_1 / A$ where A is the cross-sectional area of the wire, and $\hat{l}$ is a unit vector aligned with the axis of the wire. The volume integral $dV = Adl$, and hence the second equality follows above, where $d\mathbf{l} = \hat{l} dl$. Since $\int_{\text{Port 1}} \mathbf{E}_2 \cdot d\mathbf{l} = -V_1^{oc}$, we have the last equality above.

We can repeat the derivation with port 2 to arrive at the reaction

$$\langle \mathbf{E}_1, \mathbf{J}_2 \rangle = I_2 \int_{\text{Port 2}} \mathbf{E}_1 \cdot d\mathbf{l} = -I_2 V_2^{oc} \tag{30.3.3}$$

[7] This is the same as the current source concept in circuit theory.

Reciprocity requires these two reactions to be equal, and hence,

$$I_1 V_1^{oc} = I_2 V_2^{oc}$$

But from (30.3.1), we can set the pertinent currents to zero to find these open circuit voltages. Therefore, $V_1^{oc} = Z_{12} I_2$, $V_2^{oc} = Z_{21} I_1$. Since $I_1 V_1^{oc} = I_2 V_2^{oc}$ by the reaction concept or by reciprocity, then $Z_{12} = Z_{21}$. The above analysis can be easily generalized to an N-port network.

The simplicity of the above belies its importance. The above shows that the reciprocity concept in circuit theory is a special case of reciprocity theorem for electromagnetic theory. The terrain can also be replaced by complex circuits as in a circuit board, as long as the materials in the terrain or circuit board are reciprocal, linear and time invariant. For instance, the complex terrain can also be replaced by complex antenna structures. It is to be noted that even when the transmit and receive antennas area miles apart, as long as the transmit and receive ports of the linear system can be characterized by a linear relation expounded by (30.3.1), and the ports small enough compared to wavelength so that circuit theory prevails, we can apply the above analysis! This relation that $Z_{12} = Z_{21}$ is true as long as the medium traversed by the fields is a reciprocal medium.

Before we conclude this section, it is to be mentioned that some researchers advocate the use of circuit theory to describe electromagnetic theory. Such is the case in the transmission line matrix (TLM) method [179], and the partial element equivalence circuit (PEEC) method [180]. Circuit theory is so simple that many people fall in love with it!

30.4 Voltage Sources in Electromagnetics

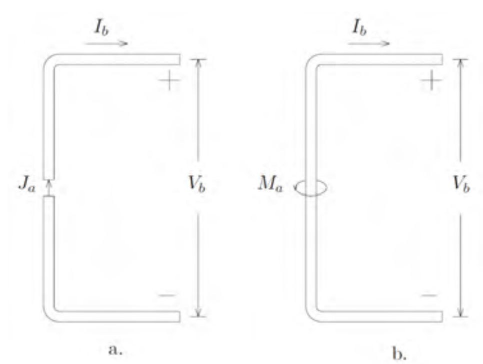

Figure 30.6: Two ways to model voltage sources: (i) An impressed current source $\mathbf{J}_a$ driving a very short antenna, and (ii) An impressed magnetic frill source (loop source) $\mathbf{M}_a$ driving a very short antenna (courtesy of Kong, Electromagnetic Wave Theory [32]).

In the above discussions, we have used current sources in reciprocity theorem to derive certain circuit concepts. Before we end this section, it is prudent to mention how voltage sources are modeled in electromagnetic theory. The use of the impressed currents so that circuit concepts can be applied is shown in Figure 30.6. The antenna in (a) is driven by a current source. But a magnetic current can be used as a voltage source in circuit theory as shown by Figure 30.6b. By using duality concept, an electric field has to curl around a magnetic current just in Ampere's law where magnetic field curls around an electric current. This electric field will cause a voltage drop between the metal above and below the magnetic current loop making it behave like a voltage source.[8]

30.5 Hind Sight

The proof of reciprocity theorem for Maxwell's equations is very deeply related to the symmetry of the operator involved. We can see this from linear algebra. Given a matrix equation driven by two different sources $\mathbf{b}_1$ and $\mathbf{b}_2$ with solutions $\mathbf{x}_1$ and $\mathbf{x}_2$, they can be written succinctly as

$$\overline{\mathbf{A}} \cdot \mathbf{x}_1 = \mathbf{b}_1 \tag{30.5.1}$$

$$\overline{\mathbf{A}} \cdot \mathbf{x}_2 = \mathbf{b}_2 \tag{30.5.2}$$

We can left dot multiply the first equation with $\mathbf{x}_2$ and do the same with the second equation with $\mathbf{x}_1$ to arrive at

$$\mathbf{x}_2^t \cdot \overline{\mathbf{A}} \cdot \mathbf{x}_1 = \mathbf{x}_2^t \cdot \mathbf{b}_1 \tag{30.5.3}$$

$$\mathbf{x}_1^t \cdot \overline{\mathbf{A}} \cdot \mathbf{x}_2 = \mathbf{x}_1^t \cdot \mathbf{b}_2 \tag{30.5.4}$$

If $\overline{\mathbf{A}}$ is symmetric, the left-hand side of both equations are equal to each other.[9] Therefore, we can equate their right-hand side to arrive at

$$\mathbf{x}_2^t \cdot \mathbf{b}_1 = \mathbf{x}_1^t \cdot \mathbf{b}_2 \tag{30.5.5}$$

The above is analogous to the statement of the reciprocity theorem which is

$$\langle \mathbf{E}_2, \mathbf{J}_1 \rangle = \langle \mathbf{E}_1, \mathbf{J}_2 \rangle \tag{30.5.6}$$

where the reaction inner product, as mentioned before, is $\langle \mathbf{E}_i, \mathbf{J}_j \rangle = \int_V \mathbf{E}_i(\mathbf{r}) \cdot \mathbf{J}_j(\mathbf{r})$. The inner product in linear algebra is that of dot product in matrix theory, but the inner product for reciprocity theorem is that for infinite dimensional spaces.[10] So if the operators in Maxwell's equations are symmetrical, then reciprocity theorem applies.

[8] More can be found in Jordain and Balmain, Electromagnetic Waves and Radiation Systems [54].

[9] This can be easily proven by taking the transpose of a scalar, and taking the transpose of the product of matrices.

[10] Such spaces are called Hilbert space.

30.6 Transmit and Receive Patterns of an Antennna

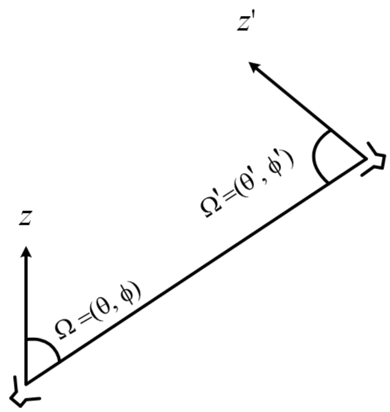

Figure 30.7: The schematic diagram for studying the transmit and receive properties of antennas. The two antennas are assumed to be identical, and each switches between transmit and receive modes in this study.

Reciprocity also implies that the transmit and receive properties of an antenna is similar to each other. The transmit property of an antenna is governed by the gain function, while its receive property is governed by the effective area or aperture. The effective aperture is also a function of angle of the incident wave with respect to to the antenna. The gain function of an antenna is related to its effective aperture by a constant as we shall argue.

Consider an antenna in the transmit mode. Then the time-average radiation power density that it will yield around the antenna, in accordance to (25.3.5), is[11]

$$\langle S_{\text{rad}} \rangle = \frac{P_t}{4\pi r^2} G(\theta, \phi) \tag{30.6.1}$$

where P_t is the total power radiated by the transmit antenna, and $G(\theta, \phi)$ is its directive gain pattern or function. It is to be noted that in the above $\int_{4\pi} d\Omega G(\theta, \phi) = 4\pi$. The above is valid when the antenna is lossless.

Effective Gain versus Directive Gain

At this juncture, it is important to introduce the concept of effective gain versus directive gain. The effective gain, also called the power gain, is

$$G_e(\theta, \phi) = f_e G(\theta, \phi) \tag{30.6.2}$$

where f_e is the efficiency of the antenna, a factor less than 1. It accounts for the fact that not all power pumped into the antenna is delivered as radiated power. For instance, power

[11]The author is indebted to inspiration from E. Kudeki of UIUC for this part of the lecture notes [139].

can be lost in the circuits and mismatch of the antenna. Therefore, the correct formula the radiated power density is

$$\langle S_{\rm rad}\rangle = \frac{P_t}{4\pi r^2} G_e(\theta,\phi) = f_e \frac{P_t}{4\pi r^2} G(\theta,\phi) \tag{30.6.3}$$

This radiated power resembles that of a plane wave when one is far away from the transmitter. Thus if a receive antenna is placed in the far-field of the transmit antenna, it will see this power density as coming from a plane wave. Thus the receive antenna will see an incident power density as

$$\langle S_{\rm inc}\rangle = \langle S_{\rm rad}\rangle = \frac{P_t}{4\pi r^2} G_e(\theta,\phi) \tag{30.6.4}$$

Effective Aperture

The effective area or the aperture of a receive antenna is used to characterize its receive property. The power received by such an antenna is then, by using the concept of effective aperture expounded in (26.1.23)

$$P_r = \langle S_{\rm inc}\rangle A_e(\theta',\phi') \tag{30.6.5}$$

where (θ',ϕ') are the angles at which the plane wave is incident upon the receiving antenna (see Figure 30.7). Combining the above formulas (30.6.4) and (30.6.5), we have

$$P_r = \frac{P_t}{4\pi r^2} G_e(\theta,\phi) A_e(\theta',\phi') \tag{30.6.6}$$

Now assuming that the transmit and receive antennas are identical. Next, we swap their roles of transmit and receive, and also the circuitries involved in driving the transmit and receive antennas. Then,

$$P_r = \frac{P_t}{4\pi r^2} G_e(\theta',\phi') A_e(\theta,\phi) \tag{30.6.7}$$

We also assume that the receive antenna, that now acts as the transmit antenna is transmitting in the (θ',ϕ') direction. Moreover, the transmit antenna, that now acts as the receive antenna is receiving in the (θ,ϕ) direction (see Figure 30.7).

By reciprocity, these two powers are the same, because $Z_{12} = Z_{21}$. Furthermore, since these two antennas are identical, $Z_{11} = Z_{22}$. So by swapping the transmit and receive electronics, the power transmitted and received will not change. A simple transmit-receive circuit diagram is shown in Figure 30.8

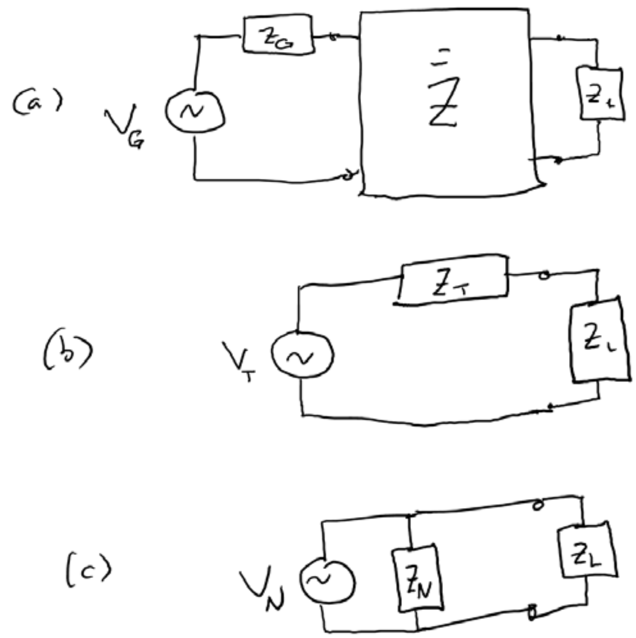

Figure 30.8: The schematic of the circuit for a transmit-receive antenna pair. Because the mutual interaction between the two antennas can be described by the impedance matrix $\overline{\mathbf{Z}}$, circuit theory can be applied to model their mutual interaction as indicated in (a). Moreover, at the receive end, one can even further simplify the circuit by using a Thevenin equivalence (b), or a Norton equivalence (c).

Consequently, we conclude that

$$G_e(\theta, \phi)A_e(\theta', \phi') = G_e(\theta', \phi')A_e(\theta, \phi) \tag{30.6.8}$$

The above implies that

$$\frac{A_e(\theta, \phi)}{G_e(\theta, \phi)} = \frac{A_e(\theta', \phi')}{G_e(\theta', \phi')} = \text{constant} \tag{30.6.9}$$

The above Gedanken experiment is carried out for arbitrary angles. Therefore, the constant is independent of angles. Moreover, this constant is independent of the size, shape, and efficiency of the antenna, as we have not stipulated their shapes, sizes, and efficiency in the above discussion.

To find this constant in (30.6.9), one can repeat the above for a Hertzian dipole, wherein the mathematics of calculating P_r and P_t is a lot simpler. This constant is found to be

$\lambda^2/(4\pi)$.[12] Therefore, an interesting relationship between the effective aperture (or area) and the directive gain function is that

$$A_e(\theta, \phi) = \frac{\lambda^2}{4\pi} G_e(\theta, \phi) \tag{30.6.10}$$

One amusing point about the above formula is that the effective aperture, say of a Hertzian dipole, becomes very large when the frequency is low, or the wavelength is very long. Of course, this cannot be physically true, and I will let you meditate on this paradox and muse over this point.

[12]See Kong [32][p. 700]. The derivation is for 100% efficient antenna. A thermal equilibrium argument is used in [139] and Wikipedia [140] as well.